INVERSE KINEMATICS

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ES 159/259

General Problem: Find $\tilde{q}$ s.t. $T_n^0(\tilde{q}) = \tilde{T}_n^0$, where $\tilde{T}_n^0$ is the desired e.e. pose.

A good approach: Write down the F.K. and try to back out the IK solution (Ex 3.8 in Spong)
and/or
Decompose robot structure into simpler pieces and project onto planes (Ex 3.10)

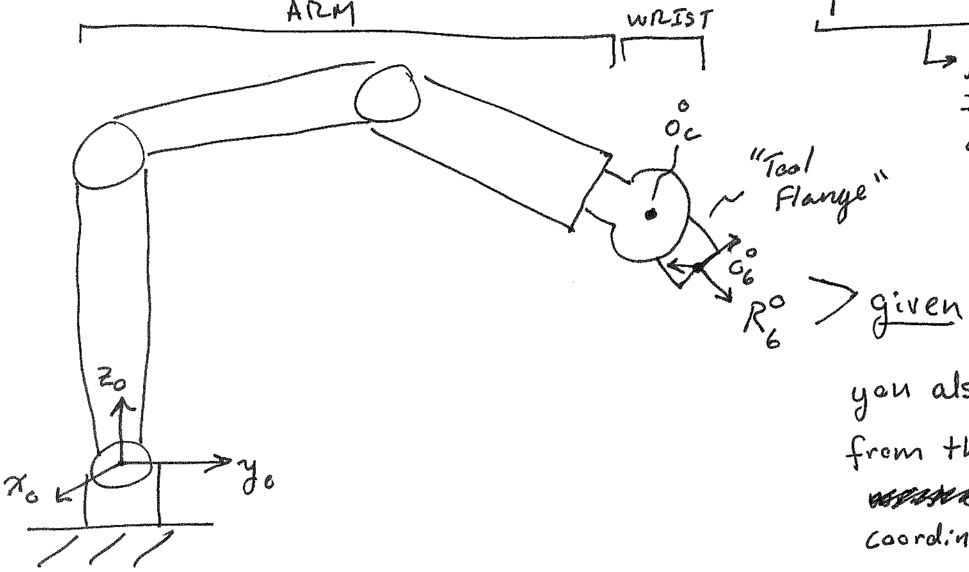
Note: This will only get you traction for simple robots!

We can find analytical solutions for an important class of robots:

6DOF manipulators consisting of a 3 DOF positioning arm & a 3 DOF spherical wrist.

↳ joint axes intersect. Intersection point is called the "wrist center" o_c^0

3 DOF ARM CAN BE RRR, PPP RRP, etc.



you also know the offset from the ~~flange~~ wrist center to the flange ~~wrist center~~ in the flange coordinate system. (from the structure of the robot)

Call this offset vector $p^6_{c,6}$. Now:

$$p^0_{c,6} = o_6^0 - o_c^0$$

& $$p^0_{c,6} = R_6^0 \, p^6_{c,6}$$

so $$o_c^0 = o_6^0 - R_6^0 \, p^6_{c,6}$$

(more general version of (3.33) & (3.34) of Spong)

STEP #1: "INVERSE POSITION KINEMATICS":

Use o_c^0 to get q_1, q_2, q_3.

Mathematically, that means:

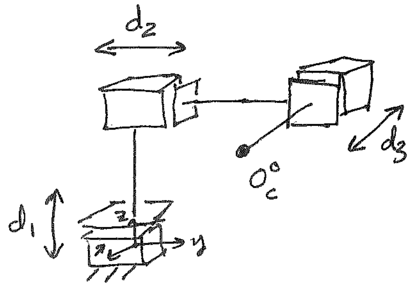
Solve $T_3^0(q_1, q_2, q_3) = \begin{bmatrix} R & o_c^0 \\ 0 & 1 \end{bmatrix}$ ← ignore for now.

$\Rightarrow$ Focus on $o_3^0(q_1, q_2, q_3) = o_c^0$

- 3 equations in 3 unknowns.
- Can invert to get analytical expression for q_1, q_2, and q_3 as functions of o_c^0
- Different # of solutions depending on robot structure & configuration.
- In practice, use decomposition & projection approach.

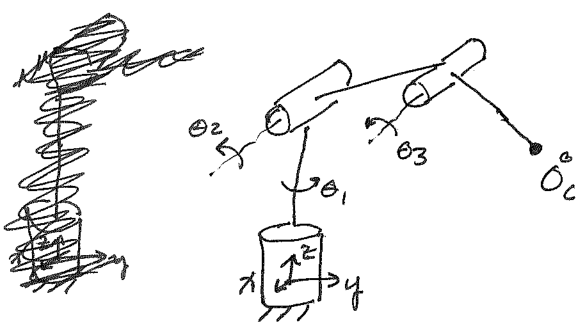
THINK THROUGH DIFFERENT CASES:

(1) PPP: HOW MANY IK SOLUTIONS?



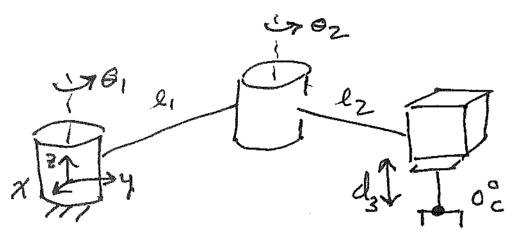
- ONE SOLUTION, GLOBALLY (assuming no joint limits)

(2) RRR:



- In most of workspace, 4 solutions
- At outer workspace boundary, 2 solutions
- When o_c^0 ~~intersects~~ lies on the joint axis of θ_1, infinite solutions

(3) RRP:



- In most of workspace, 2 solutions
- At outer workspace boundary, 1 solution
- When o_c^0 lies on the joint axis of θ_1, infinite solutions (only can occurs if $\ell_1 = \ell_2$)

Now:

STEP #2: "INVERSE ORIENTATION KINEMATICS"

- We need the wrist to correct for ~~whatever~~ the difference between the orientation of the positioning arm and the desired robot e.e. orientation.

$$\underbrace{R_6^0}_{\text{given}} = \underbrace{R_3^0}_{\substack{\text{known}\\ \text{from STEP \#1}}} \overbrace{R_6^3}^{\text{wrist orientation term}}$$

$$\Rightarrow R_6^3 = (R_3^0)^{-1} R_6^0$$

or

$$R_6^3(q_4, q_5, q_6) = (R_3^0)^{-1} R_6^0$$

FOR SPHERICAL WRISTS, A CLOSED FORM SOLUTION EXISTS! Equivalent to mapping SO(3) to Euler Angles. Sec 2.5.1 & Ex. 3.8 of Spong

SINGULARITIES

Recall: $\underset{6\times 1}{\begin{bmatrix} v \\ \omega \end{bmatrix}} = \underset{6\times n}{J}\ \underset{n\times 1}{\dot{q}}$

Column-wise:

$$\underset{6\times 1}{\begin{bmatrix} v \\ \omega \end{bmatrix}} = \left[\underset{6\times 1}{J_1}\ \ \underset{6\times 1}{J_2} \cdots \underset{6\times 1}{J_n} \right] \underset{n\times 1}{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}}$$

Remarks:
- Columns of J form a basis for all achievable e.e. velocities
- \# of independent columns determines how many e.e. velocity components we can independently select
- To achieve arbitrary $[V, \omega]$, there must be 6 independent columns of J.

IN LINEAR ALGEBRA TERMS, # of independent columns is called "rank"

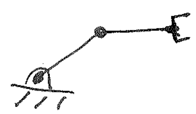
By definition, we know rank(J) is bounded by:

$$\text{rank}(J) \le \min(6, n)$$

we are particularly interested in this "<"

Any configuration in which rank(J) is less than its maximum is called a "singular configuration"

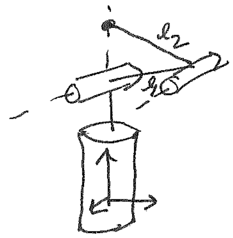
Consider 2R:



Normally Rank 2

At boundary, rank 1
- Both joints produce motion in the same direction
- J_1 & J_2 are linearly dependent

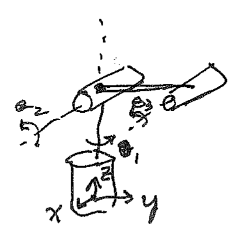
Consider RRR positioning arm: Normally RANK 3



⇐ RANK 2

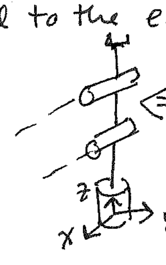
How to make RANK 1?

No motion achievable lateral to the e.e.



⇐ RANK 1 if ℓ_1 & ℓ_2 are equal

No motion in x or y!



⇐ RANK 1

No motion in z

No ~~lateral~~ motion lateral to e.e.

$v = J_v \dot{q}$ which entries are zero?

$$v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & b \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Yikes. This is a config to avoid!

SIDENOTE:

LINEAR ALGEBRA: J is singular $\iff$ J is non-invertable.

Note: In practice, it's important to know how close we are to a singular configuration.

$\Rightarrow$ One good way to do this is by looking at the condition number of J. The condition number is the ratio of the largest and smallest singular values. Even better to use the inverse condition # because it is bounded on $[1, 0]$.

SVD: SINGULAR VALUE DECOMPOSITION

$$J_{m \times n} = \overset{\text{orthogonal}}{U_{m\times m}} \Sigma_{m\times n} \overset{\text{orthogonal}}{V^T_{n \times n}}$$

$\hookrightarrow$ singular values live in this diagonal ish matrix. Describe the scaling of the system.

$$\Sigma_{m\times n} = \begin{array}{cc} \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & \ddots & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{matrix} r \\ \\ m-r \end{matrix} \\ \begin{matrix} r & \qquad n-r \end{matrix} & \end{array}$$

$r = \min(m, n)$.

This is a better ~~statement~~ than statement $r = \text{rank}(J)$ b.c.

Singular values: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$

Note greater OR Equal.

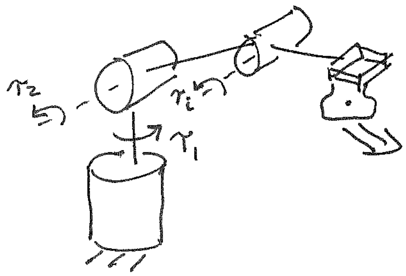
$\kappa = \sigma_1 / \sigma_r$ is condition number.

Blows up toward ∞ near singular configuration

$\left[\frac{1}{\kappa}\right]$ goes ~~to~~ toward 0 near singular configurations.

STATIC FORCE/TORQUE RELATIONSHIP:

Consider a general serial robot:



$$w_e = \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \begin{matrix} \}\text{— forces} \\ \\ \}\text{— torques} \end{matrix}$$

What joint forces/ torques $\tau_{n\times 1}$ are needed in order to apply the force/torque w_e to the environment?

Neglecting friction losses:

$$P_{in} = P_{out} \quad \text{(power conservation)}$$

Note: $F = w_e$

$$\tau_{n\times1}^T \dot{q}_{n\times1} = \dot{x}_{6\times1}^T F_{6\times1}$$

$$\tau^T \dot{q} = (J\dot{q})^T F$$

$$\dot{q}^T \tau = \dot{q}^T J^T F$$

$$\dot{q}^T(\tau - J^T F) = 0$$

$$\Rightarrow \boxed{\tau = J^T F} \quad \text{(Spong notation)}$$

or

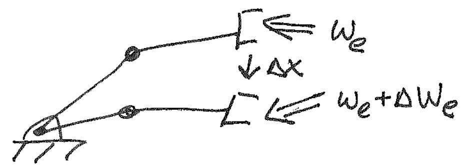
$$\boxed{\tau = J^T w_e}$$

Think about what happens to τ and F near or at singularities

- We can't control force output in all directions
- In those directions, external forces ~~do not change the ... ...~~ can be resisted with no additional joint torque.

STIFFNESS / COMPLIANCE MATRIXES

Consider a general serial robot:



ΔX: Change in e.e. pose

ΔW_e: Change in external wrench

Assume:

(1) Links are stiff compared to joints

(2) First order approximation:

ΔW_e & ΔX are small s.t. the change in J is negligable when the robot moves by ΔX.

This implies $J \Delta q = \Delta X$

$J^T \Delta W_e = \Delta \tau$

We want

$$\Delta W_e = \underbrace{K}_{\text{stiffness matrix}} \Delta X$$

$$\Delta X = \underbrace{C}_{\text{compliance matrix}} \Delta W_e$$

Let joint stiffness be represented by:

$$\Delta \tau_i = \underbrace{K_{d_i}}_{\text{individual joint stiffness}} \Delta q_i$$

individual joint stiffness with units $[N \cdot m]$ for revolute joint, $[N/m]$ for linear joint

$$\Delta \tau_{n \times 1} = K_{d_{n \times n}} \Delta q_{n \times 1}$$

$$K_d = \mathrm{diag}(K_{d_1}, K_{d_2}, \dots K_{d_n})$$

K_{d_i}: - if motor is backdrivable, will be the proportional gain of PID joint controller
- if not, then will be related to the gearhead stiffness.

Now: $$\Delta \tau = J^T \Delta W_e = K_d \Delta q$$

$$= K_d (J^{-1} \Delta X)$$

$$\underbrace{J K_d^{-1} J^T}_{C} \Delta W_e = \Delta X$$

C => compliance matrix

$$K = C^{-1} = (J^T)^{-1} K_d J^{-1}$$

↳ Stiffness matrix

$$\boxed{\begin{aligned} C &= J K_d^{-1} J^T \\ K &= (J^T)^{-1} K_d J^{-1} \end{aligned}}$$

Compliance Matrix is nice because we don't have to invert J.

In singular configs: infinite stiffness in some direction
zero compliance in that direction

- Zero is better to deal with than infinity because we can measure closeness to zero.